Threshold value of three dimensional bootstrap percolation

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The following article deals with the critical value p_c of the three-dimensional bootstrap percolation. We will check the behavior of p_c for different lengths of the lattice and additionally we will scale p_c in the limit of an infinite lattice.

Introduction

Bootstrap percolation has been motivated to describe dilute magnetic systems in which strong competition exists between magnetic and non-magnetic spins[1]. Kirkpatrick uses bootstrap percolation to discuss 3D arrays of storage servers and shows that bandwidth is the critical factor affected by the "fail in place" disorder[2].

Let each site of a d-dimensional hypercubic lattice be occupied with probability p. Then we go through each lattice site and look at all $2d$ neighboring places. A lattice site stays occupied if at least m of the $2d$ adjacent sites are occupied otherwise it becomes vacant all the time. We use a 0 for vacant and a 1 for occupied site (see [3] for a detailed description).

Let us look at the three-dimensional simple cubic lattice. We define the threshold as the value p when an infinite cluster is created. An infinite cluster is here defined as a connected set of lattice points extending from top to bottom [4]. We get p_c as function of L.

We calculate the threshold for $m = 1, .., 5$. In the case of $m = 0$ we have usual percolation. In the case of $m = 6$ lattice sites turn 0 if at least one lattice point is 0. When we turn down all lattice points at one go we have parallel updating. When we turn one after another in succession it's sequential updating. Sequential and parallel updating lead to the same results, therefore we use the sequential updating to save computing time. We have used one bit per spin and tested the results in smaller lattices at one word per spin and odd lattice size. The results were tested with different random number generators. Up to 10000 samples were made.

Results

In the case of $m = 5$ and $m = 4$ all the lattice sites become to 0 if there is no infinite cluster. Schonmann shows rigorously[5] that for all $m > d$ the

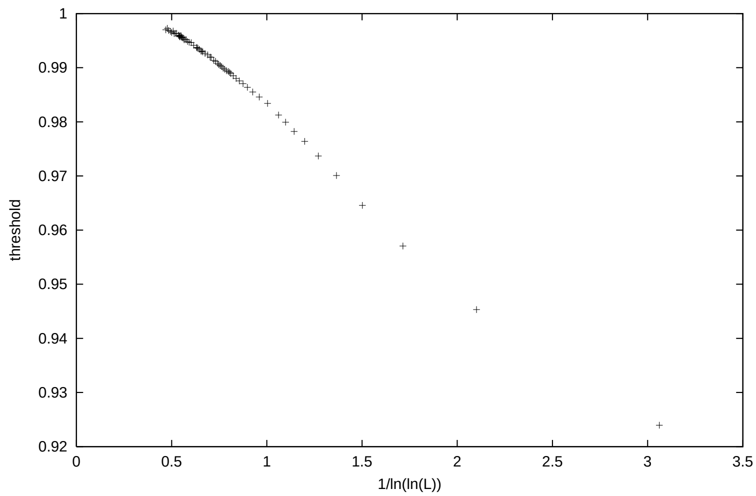

Figure 1: Bootstrap percolation $m = 5$, p_c versus $1/\ln(\ln L)$

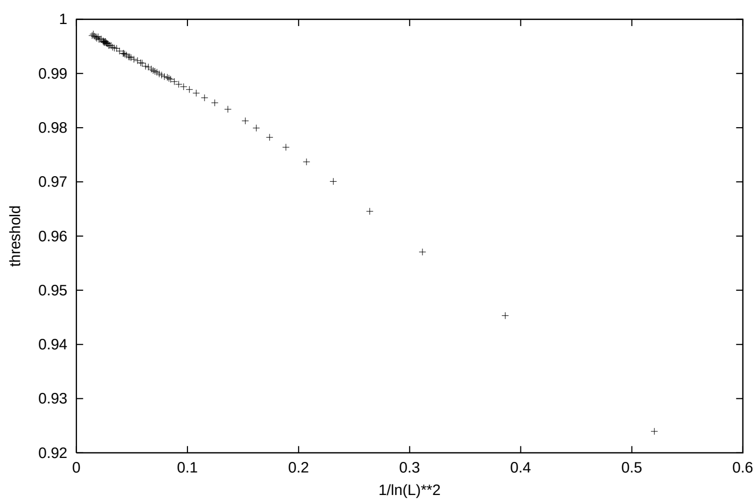

Figure 2: Bootstrap percolation $m = 5$, p_c versus $1/(\ln L)^2$

threshold for $L \to \infty$ is 1. Besides it was proven[6] that for $m = 4$, $1 - p_c$ scales as $1/\ln(\ln L)$. In agreement with earlier results[7] we obtained fig.1 to 4. In the case of $m = 5$ we have a straight line leading to 1 if we plot p_c versus $1/(\ln L)^2$ (fig.2). The theoretical prediction for $m = 4$ could not be

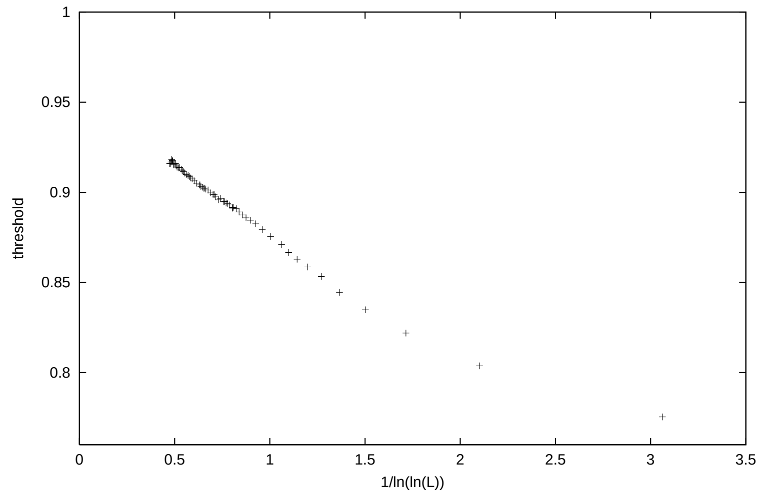

Figure 3: Bootstrap percolation $m = 4$, p_c versus $1/\ln(\ln L)$

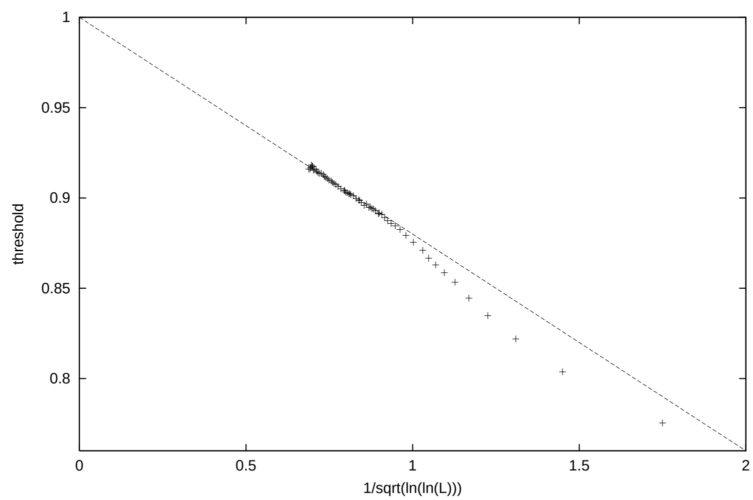

Figure 4: Bootstrap percolation $m = 4$, p_c versus $1/\sqrt{\ln(\ln L)}$

confirmed by simulations (fig.3). The best straight line $L \to \infty$ is found for p_c versus $1/\sqrt{\ln(\ln L)}$ (fig.4). We have simulated lattices from $L = 4$ up to $L = 4544$. The biggest lattice sizes were simulated on the Cray T3E parallel computer in HLRZ Jülich on up to 128 processors.

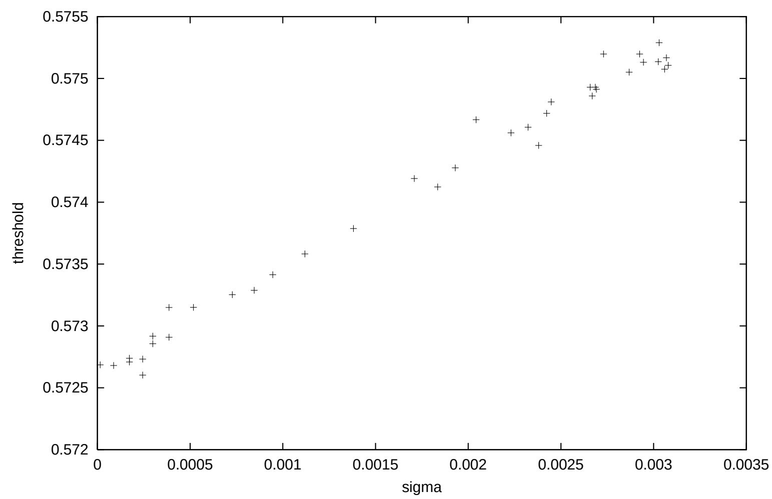

Figure 5: Bootstrap percolation $m = 3$, p_c versus σ

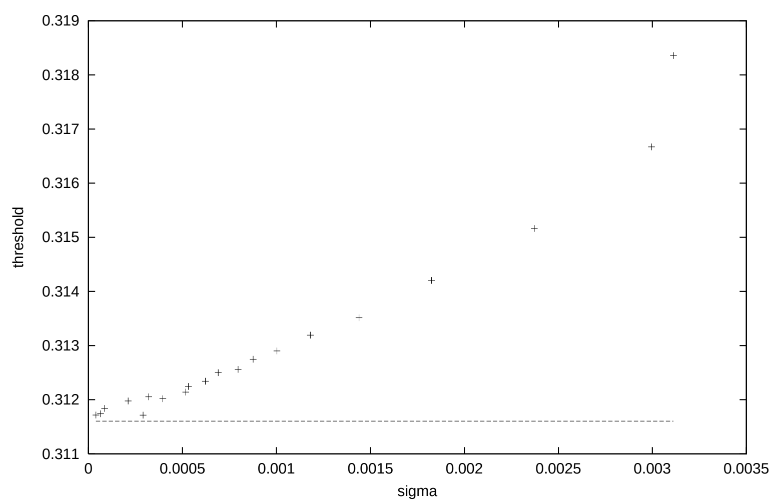

Figure 6: Bootstrap percolation $m = 2$, p_c versus σ

The case $m = 3$ is the only one that has a threshold for $L \to \infty$ between 1 and the normal threshold 0.3116[4]. When we count the occupied sites we can observe a inflection point for the fraction π of finally occupied sites as a function of the initial concentration p. But the maximal value of $d\pi/dp$ is

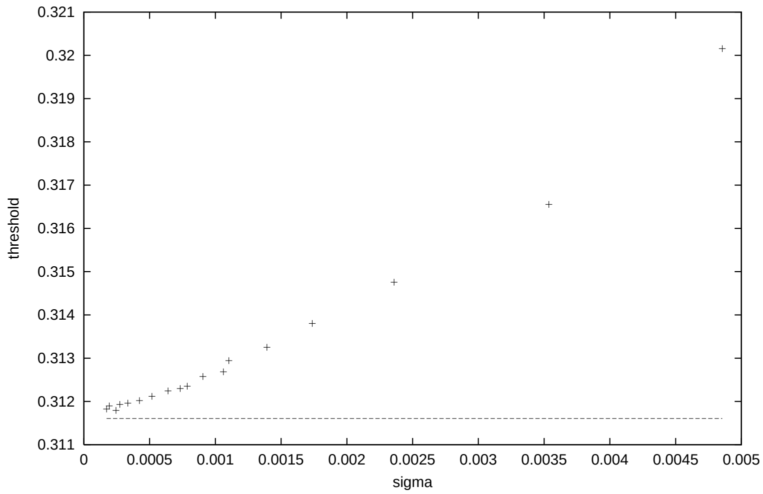

Figure 7: Bootstrap percolation m=1, p_c versus σ

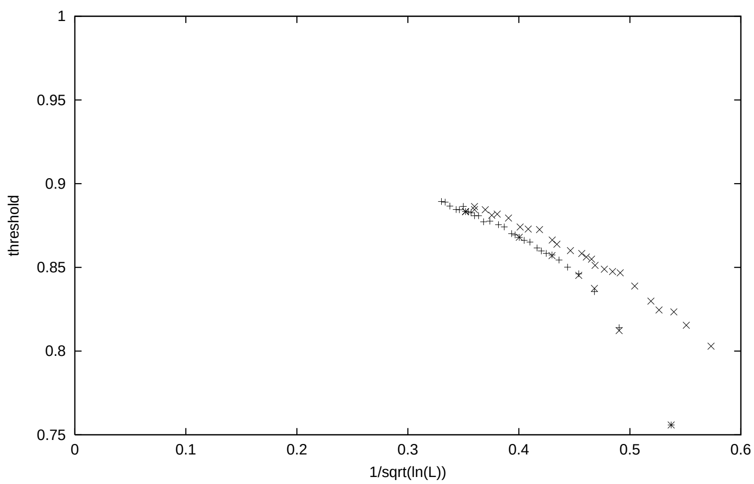

Figure 8: two-dimensional reversible bootstrap percolation $m = 3$, parallel updating, p_c versus $1/\sqrt{\ln L}$, + = one bit per spin and even lattice size, x = one word per spin with even and odd lattice sizes. There is a significant difference between odd (upper data) and even (lower data) lattice sizes. .

reached at about $p = 0.62$ for all sizes of L. Therefore the method doesn't

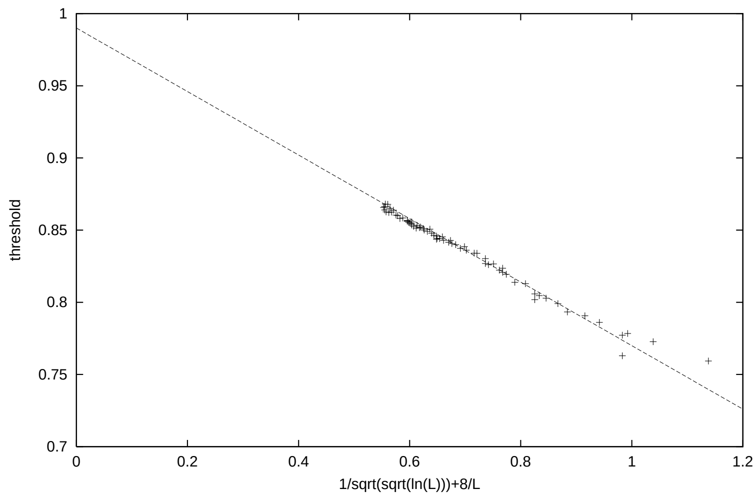

Figure 9: Two-dimensional reversible bootstrap percolation $m = 3$, sequential updating, p_c versus $(\ln L)^{-1/4} + 8/L$, $+$ = 1 bit per Spin and even lattice size, x = 1 word per spin with even and odd lattice sizes.

lead to a peak at the transition point or doesn't have a sufficiently sharp peak to useful. We must use instead the Hoshen-Kopelman algorithm[4] for cluster-analysis to find p_c. Hoshen-Kopelman have made a procedure that counts all cluster sizes[8]. When k_i is the number of clusters with i sites we get χ as $\sum_i' i^2 k_i$. We get p_{av} from the maximum of χ, and average over many samples. We know that $p_{av} - p_c \propto \sigma$ and $\sigma \propto L^{-\nu}$ with σ the width $\sigma = \sqrt{(< p_{av}^2 > - < p_{av} >^2)}$. p_{av} is the threshold for any $L < \infty$ and p_c is the threshold for infinite L. We plot p_{av} versus σ and in agreement with [9] we get a threshold of 0.5726 ± 0.0001 (fig.5). We have a slight difference between even and uneven L. The Hoshen-Kopelman algorithm prolongs computation time to about 2.2 times it's value without Hoshen-Kopelman. But computer memory limited our simulations because the larger lattice sizes need up to 1 GByte storage. The largest lattice size we have simulated is $L = 1696$, the smallest one was $L = 31$.

The cases $m = 2$ and $m = 1$ results in a value for the threshold similarly to the one expected, i.e. 0.3116 (fig. 6,7). Here the biggest lattice size is $L = 1504$ at $m = 2$ and $L = 1120$ at $m = 1$.

We have plotted only the even lattice sizes because when the lattice size is

odd the width gets larger. The reason for getting to larger L at $m = 3$ is, we can save memory by Hoshen-Kopelman because the cluster becomes rectangular.

Finally, let us have a view at the two-dimension reversible bootstrap. It also called biased majority rule model, because the state x at time $t + 1$ is a majority function of the four neighbours at time t with a bias toward occupation in case of a tie. Schonmann shows mathematically that for $m = 3$ is $C_1/\sqrt{\ln L} \leq 1 - p(L, \alpha) \leq C_2/\sqrt{\ln L}$[10,11]. When we have parallel updating this may be correct[12] but the differences between even and odd lattice size L are significant (fig.8). When we apply sequential updating that doesn't make any difference, but the critical point is lower. Sequential updating gets more empty sites by the same p so we obtained a lower threshold. When we plot p_c versus $(\ln L)^{-1/4} + 8/L$ (fig.9) we get a line extrapolated to 1.

Summary

We have independently confirmed older results even when they disagreed somewhat with theory for $m = 4$ in three dimensions, and $m = 3$ in two. Of course, the theories are valid only for $L \to \infty$ and perhaps not for our data. Our results are more accurate because we simulate bigger lattices and more lattice sizes. We get the following empirical finite-size corrections:

critical behavior p_c of the three-dimensional bootstrap percolation			
m	p_c for $L \to \infty$	versus	rigorously
5	1	$1/(\ln L)^2$	
4	1	$1/\sqrt{\ln(\ln L)}$	$1/\ln(\ln L)$
3	0.5726 ± 0.0001	σ	
2	0.3116	σ	0.3116
1	0.3116	σ	0.3116
0	0.3116	σ	0.3116
two-dimensional bootstrap percolation			
3	1	$(\ln L)^{-1/4}$	$(\ln L)^{-1/2}$

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